

Quantum Hackathon

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New to quantum mechanics? Then start with our Dirac notation boot camp in Section [A](#).

*<https://www.qotlabs.org/en/index.php>

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1 Quantum black-box

One of the main problems with all algorithms, quantum not excluded, is that they not always perform as promised. In that case you need to understand where the problem is and try to fix it. Let's focus on the first part. In this problem you will face an unknown single-qubit gate, which we call a quantum black box and try to estimate, what it does.

1.1 Qubit states

If you have followed Section A carefully, you now know that quantum states are vectors in Hilbert spaces. For example, for a single qubit this vector space spans all linear combinations of two basis vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1)$$

So any qubit state is a vector

$$|\psi\rangle = c_1 |0\rangle + c_2 |1\rangle \quad (2)$$

with arbitrary complex coefficients c_1, c_2 , satisfying the normalization condition

$$|c_1|^2 + |c_2|^2 = 1.$$

To get some geometrical intuition it is instructive to visualize a qubit state as a point on the two-dimensional sphere. Indeed, the overall phase of the complex vector does not matter, so we can identify all vectors which differ only in phase: $|\psi\rangle \simeq e^{i\alpha} |\psi\rangle$. That means, we can rewrite (2) using two real parameters only:

$$|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle. \quad (3)$$

Then you can identify θ with a polar angle, and ϕ with an azimuthal angle on a sphere. Notice, that orthogonal states correspond to diametrically opposite points on the sphere.

1.2 Unitary evolution

The evolution of every isolated quantum system (we did our best to isolate our black box from the environment) is governed by a unitary transformation. So if the input state is $|\psi_{\text{in}}\rangle$, then the output state would be

$$|\psi_{\text{out}}\rangle = U |\psi_{\text{in}}\rangle, \quad (4)$$

where U is a unitary matrix ($U^\dagger U = U U^\dagger = 1$). The matrix has to be unitary to preserve vector norm and inner product. For a single-qubit transformations, a.k.a. *single-qubit gates*, U is a 2×2 unitary matrix.

1.3 Quantum measurements

Quantum measurements are your means of communication with the quantum world and the only way to extract information from the quantum computer. The problem is that they are intrinsically probabilistic. So the outcomes of your measurements will be samples from some probability distribution. Fortunately, quantum theory tells us exactly, how this probability distribution will look like.

If the measurement has d outcomes, then the probability distribution is $\{p_0, p_1, \dots, p_{d-1}\}$. Every outcome k is identified with a positive operator M_k in such a way, that the *Born's rule* holds:

$$p_k = \langle \psi | M_k | \psi \rangle. \quad (5)$$

Measurement operators must satisfy $\sum_k M_k = 1$. We will deal with the simplest case, when the measurement operators M_k are one-dimensional projectors:

$$M_k = |\psi_k\rangle\langle\psi_k|, \quad (6)$$

in this case we call them *projective measurements*. This is the only type of measurements, that our quantum computer allows you to do.

Now we are only dealing with qubits today. In this case every projective measurement may have only two outcomes with probabilities $\{p, 1 - p\}$. The corresponding projectors will be

$$M_0 = |\psi\rangle\langle\psi|, \quad M_1 = 1 - |\psi\rangle\langle\psi| = |\psi^\perp\rangle\langle\psi^\perp|, \quad (7)$$

where $\langle\psi|\psi^\perp\rangle = 0$.

1.4 The problem

Now we are ready to formulate the problem. You are given access to a quantum black box. It is an elementary quantum computer, which takes a single qubit input and outputs a single qubit. We know for sure, that it performs some unitary transformation U , but we do not know, which one. You are free to feed it with any single qubit states and make any projective single-qubit measurements on the output. Your task is to determine the unitary matrix U as precise, as possible.

You will be able to measure 6 processes divided into two groups: the **test** group and the **competition** group. For processes in the test group you will have an unlimited number of trials, such that you can practice and develop the best algorithm. For processes in the competition group the number of trials will be limited to 10^4 .

1.5 Communication with the black box

Software, allowing you to communicate with the black box, is implemented as a server application. The server address and port is `ape.qotlabs.org:2018`. You should send requests to this server in order to obtain results. An example Python 3 script, which performs a single request to the server, is presented in the listing 1.

```

1 #!/usr/bin/python3
2 import socket, sys
3
4 request='password 1 100 0.5 0.5 0.5 0.5 1 0 0 0'
5
6 host = 'ape.qotlabs.org'
7 port = 2018
8
9 # connect to remote host
10 s = socket.socket(socket.AF_INET, socket.SOCK_STREAM)
11 s.settimeout(2)
12 try :
13     s.connect((host, port))
14 except :
15     print('Unable to connect to ',host,':',port)
16     sys.exit()
17
18 # connected to remote host
19 s.send(request.encode())
20 data = s.recv(4096).decode()
21 print (data)

```

Listing 1: Example Python script, which performs one request to the server.

Each request is a string consisting of 11 whitespace-separated fields:

```
'team_password process_index number_of_states s0 s1 s2 s3 m0 m1 m2 m3'
```

1. `team_password` – a password given to each team by the organizers that grants access to the server. The password must be kept in secret from other teams!
2. `process_index` – the index of the unitary process to simulate. Available values are $0, \dots, 5$. Processes with indices $0, 1, 2$ belong to the test group and processes $3, 4, 5$ belong to the competition group. While there are no restrictions for the test group, the accumulated `number_of_states` for each competition process is limited by 10^4 . If the limit is exceeded the server will return error message “Limit for this process exceeded!”. The performance of your tomography algorithm is evaluated only for the competition group, so you must submit the results for this group (indices $3, 4, 5$) and not for the test one.

Unitary matrices of the test processes are:

$$U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (8)$$

$$U_1 = \begin{pmatrix} -0.5312911563 - 0.02726422414i & -0.4179109327 - 0.7364352122i \\ 0.7838199623 - 0.3203320715i & -0.4343945285 - 0.3071075125i \end{pmatrix}, \quad (9)$$

$$U_2 = \begin{pmatrix} 0.9190171056 - 0.1703624309i & 0.3554755347 - 0.004620165052i \\ 0.2707565243 - 0.2303803513i & -0.8042113453 + 0.4762981319i \end{pmatrix}. \quad (10)$$

3. `number_of_states` – the number of input states to pass through the process and measure.
4. `s0 s1 s2 s3` – parameters of the input state. The vector $|\psi_{\text{in}}\rangle$ is expressed as follows:

$$|\psi_{\text{in}}\rangle = \begin{pmatrix} s_0 + is_1 \\ s_2 + is_3 \end{pmatrix}. \quad (11)$$

You should ensure that the input vector is properly normalized $\langle \psi_{\text{in}} | \psi_{\text{in}} \rangle = 1$. The server tests this constraint within tolerance `tol = 1e-7`, so we recommend to provide at least 8 digits of precision, when you generate a request. If the constraint is not fulfilled, the corresponding error message is returned.

5. `m0 m1 m2 m3` – parameters of the vector $|\psi\rangle$, corresponding to the projective measurement operator M_0 (see eq. (7)), in the form analogous to the input state specification (see eq. (11)). The second operator M_1 is calculated by our software: $M_1 = 1 - M_0$. You must ensure that the measurement vector is normalized (if it is not, then the server returns error).

On success, the server returns a string containing two integers n_0 and n_1 , separated by a whitespace. These quantities are, respectively, number of registered outcomes for measurements M_0 and M_1 after `number_of_states` has been measured. Their sum is $n_0 + n_1 = \text{number_of_states}$, since probabilities p_0 and p_1 sum to unity, $p_0 + p_1 = 1$.

1.6 Submission details

You should submit your estimated unitary matrices U_i for processes with indices $i = 3, 4, 5$ as separate files, titled according to a template `team_name_process_i.dat`. Each file should contain

real and imaginary parts of matrix elements, separated by whitespaces, according to the following format:

$$\begin{array}{cccc} \text{Re } U_{11} & \text{Im } U_{11} & \text{Re } U_{12} & \text{Im } U_{12} \\ \text{Re } U_{21} & \text{Im } U_{21} & \text{Re } U_{22} & \text{Im } U_{22} \end{array} \quad (12)$$

For example, the identity matrix, written in this format, is

$$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \quad (13)$$

Since we test unitarity of matrices within tolerance `tol = 1e-7`, we recommend to provide at least 8 digits of accuracy. The files should be sent to hackathon@qotlabs.org. Additionally, you must provide a team password in the message.

2 3-SAT problem and QAOA

Propositional satisfiability (SAT) is the problem of determining the satisfiability of sentences in propositional logic. If k is the number of literals in each clause the problem is called K -SAT. Determining the satisfiability of a formula in conjunctive normal form where each clause is limited to at most k literals is NP-complete (Cook [2]). Every K -SAT problem can be reduced to 3-SAT problem:

$$(x_i \vee \bar{x}_j \vee x_k) \wedge (\bar{x}_g \vee x_r \vee \bar{x}_s) \wedge (\dots) \wedge \dots$$

For example the following 3-Sat problem

$$(x_1 \vee \bar{x}_2 \vee x_4) \wedge (x_1 \vee x_3 \vee \bar{x}_5) \quad (14)$$

is satisfiable for $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0, x_5 = 0$.

It is possible to embed this problem into the low-energy spectrum of a Hermitian operator formed by the linear extension of $\{P_0, P_1, \mathbb{1}\}$ by means of the following invertible mapping:

$$\wedge \longrightarrow \otimes, \quad \vee \longrightarrow +. \quad (15)$$

And for boolean variables $x_j, \neg x_j$:

$$x_j \longrightarrow P_0^j, \quad \neg x_j \longrightarrow P_1^j. \quad (16)$$

Indeed, the first clause of (14) is equal to 0 only one bit-string 010 and the clause correspond to a Hamiltonian:

$$\mathcal{K} = P_0^1 \otimes P_1^2 \otimes P_0^4,$$

and for all bit-strings the eigenvalues of \mathcal{K} is equal to 0 except $|010\rangle$ which has eigenvalue 1.

Further reading [1],[5],[4].

2.1 QAOA

Let us consider an cost function of some optimization problem defined on n bits and m clauses (for example 3-SAT problem) – the number of unsatisfiable clauses

$$\mathcal{H}(x) = \sum_{i=1}^m h_i(x) \quad (17)$$

such that $h_i(x) = 0$ if the clause is satisfiable, otherwise it is equal to 1.

Suppose we have a quantum computer which works in 2^n dimensional Hilbert space with computational basis $|x\rangle$ of bits and that quantum version of our optimization problem i.e. Hamiltonian is diagonal in this basis

$$\mathcal{H} = \sum_{i=1}^m h_i. \quad (18)$$

Our goal is to find the ground state of \mathcal{H} .

For this purpose we define unitary operators:

$$U_{\mathcal{H}}(\alpha) = e^{i\gamma H} = \prod_{j=1}^m e^{i\gamma h_j}, \quad U_{\mathcal{D}}(\beta) = e^{i\beta D} = \prod_{j=1}^n e^{-i\beta X_j},$$

$$D = -\sum_{j=1}^n X_j$$

and quantum state for any integer p

$$|\alpha, \beta\rangle = U_{\mathcal{D}}(\beta_p)U_{\mathcal{H}}(\alpha_p) \dots U_{\mathcal{D}}(\beta_1)U_{\mathcal{H}}(\alpha_1) |+\rangle^{\otimes n} \quad (19)$$

Then the expectation value of \mathcal{H} in this state is:

$$C_p(\alpha, \beta) = \langle \alpha, \beta | \mathcal{H} | \alpha, \beta \rangle \quad (20)$$

It can be shown that

$$\min_{\psi} \frac{\langle \psi | \mathcal{H} | \psi \rangle}{\langle \psi | \psi \rangle} = \lim_{p \rightarrow \infty} \min_{\alpha, \beta} C_p(\alpha, \beta) \quad (21)$$

That is why on a quantum computer we can do the following procedure [3] to find the solution of the optimization problem:

1. Pick a number of iterations p
2. Run a quantum computer for different angles α, β on a grid $[0, 2\pi]^p \times [0, \pi]^p$ to find the minimum of C_p

2.2 Problems

Since MSU Quantum Computer will be busy today operating as a single-qubit black box we suggest the following problems. Calculate parameters α, β and overlap between prepared state and ground state space on your classical laptop (i.e. simulate quantum computer on your laptop) for the following 3-SAT problems and number of iterations p

1. $(x_1 \vee x_2 \vee \neg x_6) \wedge (x_1 \vee \neg x_4 \vee \neg x_5) \wedge (x_1 \vee x_3 \vee \neg x_6) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_6) \wedge (x_4 \vee x_5 \vee \neg x_6)$ and $p = 20$
2. $(\neg x_2 \vee x_4 \vee x_6) \wedge (\neg x_6 \vee \neg x_7 \vee x_8) \wedge (\neg x_4 \vee x_7 \vee \neg x_8) \wedge (x_2 \vee x_3 \vee \neg x_6) \wedge (x_1 \vee x_6 \vee \neg x_7) \wedge (\neg x_1 \vee x_6 \vee \neg x_8) \wedge (\neg x_2 \vee x_5 \vee x_6) \wedge (x_1 \vee \neg x_5 \vee \neg x_8)$ and $p = 12$
3. $(\neg x_1 \vee x_4 \vee x_6) \wedge (\neg x_2 \vee \neg x_3 \vee x_5) \wedge (\neg x_7 \vee x_{10} \vee \neg x_{11}) \wedge (x_8 \vee x_9 \vee \neg x_{12}) \wedge (x_{17} \vee x_{13} \vee \neg x_{15}) \wedge (\neg x_{14} \vee x_{20} \vee \neg x_{18}) \wedge (\neg x_{16} \vee x_{19} \vee x_{21})$ and $p = 5$

2.3 Submission details

For problems 1,2 return list of parameters $\alpha_1, \beta_1, \dots, \alpha_p, \beta_p$ as a csv file with comma separator as team_name_qaoa_1.csv and team_name_qaoa_2.csv where team_name is the name of your team and send it to hackathon@qotlabs.org.

	result
0	0.570918
1	0.854578
2	1.031475
3	0.478164
4	1.657587
5	0.260604
6	0.687500

Figure 1: Submission file. First rows are parameters $\alpha_1, \beta_1, \dots, \alpha_p, \beta_p$ and the last row is an overlap between prepared state and ground state space

For the last problem send a program to hackathon@qotlabs.org

References

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A Dirac Notation boot camp

Let ψ and ϕ be vectors in a n -dimensional complex vector space \mathcal{V} . We adopt Dirac notation in which

$$\psi \rightarrow |\psi\rangle, \quad \phi \rightarrow |\phi\rangle$$

and an inner product of these states is $\langle\psi|\phi\rangle$. We assume that inner product is linear in the second argument and antilinear in the first, i.e.

$$\begin{aligned} \langle\psi|\alpha\phi_1 + \beta\phi_2\rangle &= \alpha\langle\psi|\phi_1\rangle + \beta\langle\psi|\phi_2\rangle, \\ \langle\alpha\psi_1 + \beta\psi_2|\phi\rangle &= \bar{\alpha}\langle\psi_1|\phi\rangle + \bar{\beta}\langle\psi_2|\phi\rangle, \end{aligned}$$

where $\alpha, \beta \in \mathbb{C}$. Every vector $|\psi\rangle \in \mathcal{V}$ can be expanded in the orthonormal basis $\{|i\rangle\}$:

$$|\psi\rangle = \sum_i \alpha_i |i\rangle, \quad \alpha_i = \langle i|\psi\rangle \in \mathbb{C}.$$

Let A be a linear operator $A : \mathcal{V} \rightarrow \mathcal{V}$. In Dirac notations linear means

$$A(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha A|\psi\rangle + \beta A|\phi\rangle.$$

Instead of $A|\psi\rangle$ we sometimes use $|A\psi\rangle$ or $|A(\psi)\rangle$.

Remark A.1. (Linearity of the bracket). As is typical in many schools of quantum physics, we observe linearity of the bracket as $|A(\phi)\rangle = A|\phi\rangle$ and $|a \cdot \phi + b \cdot \psi\rangle = a \cdot |\phi\rangle + b \cdot |\psi\rangle$.

Let us define projectors onto unit vectors $|\psi\rangle$ as

$$P_\psi \stackrel{\text{def}}{=} |\psi\rangle\langle\psi|. \tag{22}$$

It is a projector as $P_\psi^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = P_\psi$. Moreover, every Schmidt rank-1 projector in \mathcal{V} can be represented in form (22).

Problem 1. (Rank-1 projectors \star). A non-trivial operator P is a Schmidt rank-1 projector if and only if it can be written as $|\psi\rangle\langle\psi|$.

Solution. Let us write P in terms of its singular value decomposition $P = UDV^\dagger$. Assume P is a projector. Then $P^2 = P$ implies that

$$UDV^\dagger UDV^\dagger = UDV^\dagger \tag{23}$$

and so $V^\dagger U = \mathbf{1}$ and hence $V = U$. We further have from (23) that $D^2 = D$ and so $\text{spec}\{D\} \in \{0, 1\}$. For D rank-1, only a single eigenvalue is non-zero. The eigenvalues of D are monotonically ordered lexicographically in the standard basis. Hence $D = |\mathbf{0}\rangle\langle\mathbf{0}|$ and so $U|\mathbf{0}\rangle\langle\mathbf{0}|U^\dagger = |\psi\rangle\langle\psi|$. Note that an example of projector that has full rank k is the $k \times k$ identity. One with rank $k - 1$ is found considering $\mathbf{1} - |a\rangle\langle a|$.

For every vector in orthonormal basis $\{|i\rangle\}$ we define a projector $P_i = |i\rangle\langle i|$ then

$$\sum_i P_i |\psi\rangle = \sum_i |i\rangle\langle i| \sum_j \alpha_j |j\rangle = \sum_i \alpha_i |i\rangle = |\psi\rangle,$$

i.e.

$$\mathbb{1} = \sum_i P_i,$$

where $\mathbb{1}$ is the identity operator on \mathcal{V} .

Let \mathcal{W} be a m -dimensional complex vector space. The vector $|\psi\rangle \otimes |\phi\rangle \in \mathcal{V} \otimes \mathcal{W}$, where $|\psi\rangle \in \mathcal{V}$, $|\phi\rangle \in \mathcal{W}$. In Dirac notation it is

$$|\psi\rangle |\phi\rangle = |\psi, \phi\rangle \stackrel{\text{def}}{=} |\psi\rangle \otimes |\phi\rangle.$$

If A is a linear operator $A : \mathcal{V} \rightarrow \mathcal{V}$ and B is a linear operator $B : \mathcal{W} \rightarrow \mathcal{W}$ then

$$A \otimes B : \mathcal{V} \otimes \mathcal{W} \rightarrow \mathcal{V} \otimes \mathcal{W}, \quad A|\psi\rangle \otimes B|\phi\rangle = |A\psi\rangle |B\phi\rangle. \quad (24)$$

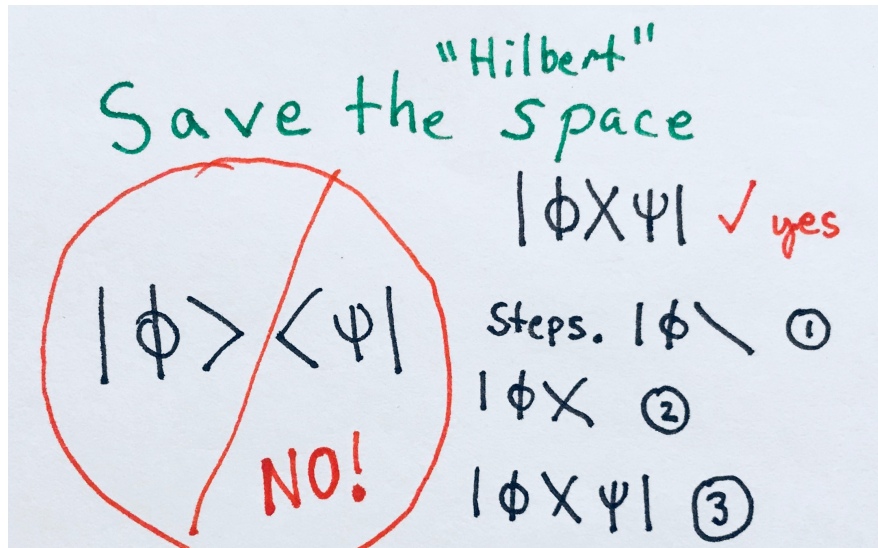


Figure 2: Lifhack: how to save space using Dirac notation.

A.1 Example: \mathbb{C}^2

As an example let us consider vector space \mathbb{C}^2 with orthogonal basis vectors $|0\rangle, |1\rangle$. A general state in \mathbb{C}^2 can then be written as

$$|\psi\rangle = \cos(\theta) |0\rangle + e^{-i\phi} \sin(\theta) |1\rangle \quad (25)$$

and we recall the Pauli operators defined by the following formulas

$$\begin{aligned} \mathbb{1} &= |0\rangle\langle 0| + |1\rangle\langle 1| = \sum |a\rangle\langle a|, & X &= |0\rangle\langle 1| + |1\rangle\langle 0| = \sum |1-a\rangle\langle a|, \\ Y &= -i|0\rangle\langle 1| + i|1\rangle\langle 0| = i \sum (-1)^a |a\rangle\langle 1-a|, & Z &= |0\rangle\langle 0| - |1\rangle\langle 1| = \sum (-1)^a |a\rangle\langle a| \end{aligned}$$

for $a \in \{0, 1\}$.

Hence if $\alpha |0\rangle + \beta |1\rangle$ is an eigenvector of X and λ is the corresponding eigenvalue then

$$X(\alpha |0\rangle + \beta |1\rangle) = \alpha |1\rangle + \beta |0\rangle = \lambda(\alpha |0\rangle + \beta |1\rangle).$$

And we come to the system of equations $\lambda\alpha = \beta$ and $\lambda\beta = \alpha$, which has two solutions $\lambda = \pm 1$. Consequently eigenvectors of X are

$$|+\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

Problem 2. By using Dirac notation (i) find eigenvalues and eigenvectors of Y, Z . (ii) Show that

$$[X, Y] = 2iZ, \quad [Y, Z] = 2iX, \quad [Z, X] = 2iY,$$

and hence (iii \star) show that $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$, where ϵ is the Levi-Civita symbol, $\sigma_1 = X, \sigma_2 = Y$ and $\sigma_3 = Z$.

Projectors on $|0\rangle, |1\rangle$ states in Dirac notations are

$$P_0 \stackrel{\text{def}}{=} |0\rangle\langle 0| = \frac{1}{2}(\mathbf{1} + Z), \quad P_1 \stackrel{\text{def}}{=} |1\rangle\langle 1| = \frac{1}{2}(\mathbf{1} - Z).$$

Projector on arbitrary state $|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$

$$P_\psi \stackrel{\text{def}}{=} |\psi\rangle\langle\psi| = (\alpha|1\rangle + \beta|0\rangle)(\alpha\langle 1| + \beta\langle 0|) = |\alpha|^2|1\rangle\langle 1| + \alpha\bar{\beta}|1\rangle\langle 0| + \bar{\alpha}\beta|0\rangle\langle 1| + |\beta|^2|0\rangle\langle 0|.$$

For example projector on $|+\rangle$ state in terms of ket-bra vectors of $\{0, 1\}$ is equal to

$$|+\rangle\langle +| = \frac{1}{2}(|1\rangle\langle 1| + |1\rangle\langle 0| + |0\rangle\langle 1| + |0\rangle\langle 0|).$$

Problem 3. Define iff statements on condition on the a_{ij} terms for the operator

$$\sum_{i,j \in \{0,1\}} a_{ij} |i\rangle\langle j| \tag{26}$$

to be (i) Hermitian and (ii) a projector.

Let us consider the tensor product of two eigenvectors of X operator

$$|+-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle),$$

this state is in $\mathbb{C} \otimes \mathbb{C}$. In this space we can define so-called Bell states with entanglement property (see Problem 5)

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad |\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle).$$

Problem 4. (1) Show that the corresponding eigenvectors of the Pauli matrices X, Y, Z are $\sqrt{2}|\pm\rangle = |0\rangle \pm |1\rangle$ and $\sqrt{2}|y_\pm\rangle = |0\rangle \pm i \cdot |1\rangle$ and $|j\rangle$ for $j = 0, 1$.

(2) Consider the operators $X \otimes X, Y \otimes Y, Z \otimes Z$. Use the tensor product to define an eigenbasis and calculate the eigenvalues for these same operators.

(★) Find an eigenbasis for these same operators which can not be factorized using the tensor product. (In quantum theory, these basis states are called, entangled.)

Problem 5. Show that $\nexists |\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^2$ for the Bell state to be expressed as $|\Phi^\pm\rangle = |\psi_1\psi_2\rangle$. The states $|\Phi^\pm\rangle$ are therefore called entangled.

Problem 6. (Bell basis). Using the right hand side equations,

$$\begin{aligned}\sigma_0 &= \sum |a\rangle\langle a|, & \sigma_1 &= \sum |1-a\rangle\langle a|, \\ \sigma_2 &= i \sum (-1)^a |a\rangle\langle 1-a|, & \sigma_3 &= \sum (-1)^a |a\rangle\langle a|.\end{aligned}$$

for $a \in \{0, 1\}$. Show that

$$\frac{1}{\sqrt{2}} \left(\sum_{l \in \{0,1\}} \langle l| \otimes \langle l| \right) \sigma_i \otimes \mathbf{1} = \frac{1}{\sqrt{2}} \sum_{l \in \{0,1\}} (\langle l| \sigma_i) \otimes \langle l| \quad (27)$$

defines the Bell states (which defines an orthonormal basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$). Here σ_i indexes the Pauli matrices.

Problem 7. Show that $P_1 \otimes U + P_0 \otimes \mathbf{1}$ is unitary if and only if U is unitary.

Problem 8. Show that $A^\dagger B^\dagger = B^\dagger A^\dagger$ if and only if $[A, B] = 0$.